

Centrality, Noncentral Selection, and the Scope of Bell–CHSH: A Measure-Theoretic Critique, Sharp Inflation Bounds, Canonical Reweighting Constructions, and an Experimental Protocol

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Abstract

Bell–CHSH is frequently paraphrased as “local realism is impossible” after experimental violations of CHSH. The theorem is correct, but the paraphrase often suppresses a structural assumption: *selection centrality* (fair sampling), i.e. that the detected sample is not a setting- and hidden-variable-dependent reweighting of the hidden prior. We give a measure-theoretic separation between (i) the *central* sector, where $E_{\text{obs}} = E_{\text{full}}$ and $\text{CHSH} \leq 2$ holds under measurement independence and locality, and (ii) the *non-central* sector, where the observed correlations are computed under a reweighted measure $d\nu_{a,b} = w(a, b, \lambda) d\rho$. We prove a sharp CHSH inflation bound

$$S_{\text{obs}} \leq 2 + 4\delta, \quad \delta := \sup_{a,b} \int |w(a, b, \lambda) - 1| \rho(d\lambda),$$

so reproducing Tsirelson’s $2\sqrt{2}$ by strictly local measurement-independent models via selection requires $\delta \geq (2\sqrt{2} - 2)/4 \approx 0.2071$. We provide (A) a canonical Radon–Nikodým construction (piecewise constants on sign regions plus a small positive remainder), (B) an explicit local factorized detection-loophole construction that reproduces arbitrary finite correlation tables, and (C) an implementable optics protocol to estimate or upper-bound δ from time-tag data via window dithering, threshold and spectral sweeps, and auxiliary-tag binning. This paper critiques not the validity of Bell’s theorem, but the common overstatement of what CHSH violations imply without an explicit, quantitative centrality audit.

Keywords: Bell–CHSH inequality, Bell’s theorem, local hidden variables (LHV), measurement independence (MI), locality, realism, fair sampling, central selection, noncentral selection, detection loophole, post-selection, coincidence window, selection bias, Radon–Nikodým derivative, reweighted measure, total variation distance, L^1 deviation, CHSH inflation bound, Tsirelson bound, singlet correlations, correlation kernel, bipartite correlations, outcome functions $A(a, \lambda), B(b, \lambda)$, detection efficiencies η_A, η_B , acceptance gates, auxiliary-tag binning, bootstrap confidence interval, threshold sweep, spectral filter sweep, time-tag analysis

Contents

1	Introduction (what is being critiqued)	2
2	Framework	2
2.1	Local outcomes	3
2.2	Local detection and selection	3
2.3	Observed vs. unconditional correlations	3
2.4	A quantitative noncentrality statistic	3
3	The central sector: where Bell–CHSH actually lives	4
4	Noncentral selection: sharp inflation bounds	4
5	Canonical RN-density recipe (measure-theoretic)	5
6	Explicit local factorized construction for arbitrary finite correlation tables	5
7	PV-style motivation: fibered noninjectivity (optional)	6
8	Experimental protocol to estimate/bound δ	6
9	Conclusion	8
A	Appendix: a common mistaken “local model” and the correct integral	8

1 Introduction (what is being critiqued)

Bell’s theorem is mathematically correct: under measurement independence (MI), local response functions, and bounded outcomes, the *unconditional* correlations obey $\text{CHSH} \leq 2$. Modern experiments violate CHSH and therefore exclude the *central* local hidden-variable class.

The critique addressed here is narrower and operational:

CHSH violation implies nonlocality *only* after one has controlled/justified that the *observed* correlation equals the *unconditional* correlation, i.e. that selection is *central* (fair sampling) or that any deviation from centrality is bounded below the amount needed to inflate CHSH above 2.

In any real experiment, “the observed sample” is produced by hardware gates, thresholds, timing rules, coincidence identification, and analysis windows. All of these can be expressed as a setting- and hidden-variable-dependent selection weight.

2 Framework

Let S_A, S_B be setting sets for Alice and Bob. Let $(\Lambda, \mathcal{F}, \rho)$ be a probability space of hidden variables λ . *Measurement independence* (MI) means ρ does not depend on (a, b) .

2.1 Local outcomes

Outcomes are bounded and local:

$$A(a, \lambda) \in [-1, 1], \quad B(b, \lambda) \in [-1, 1],$$

with A depending only on (a, λ) and B only on (b, λ) .

2.2 Local detection and selection

Let $\eta_A(a, \lambda), \eta_B(b, \lambda) \in [0, 1]$ be detection probabilities (or acceptance gates). Define

$$Z(a, b) := \int \eta_A(a, \lambda) \eta_B(b, \lambda) \rho(d\lambda) \in (0, 1],$$

and the normalized selection weight

$$w(a, b, \lambda) := \frac{\eta_A(a, \lambda) \eta_B(b, \lambda)}{Z(a, b)}, \quad \int w(a, b, \lambda) \rho(d\lambda) = 1. \quad (1)$$

Equivalently, define the detected-sample measure $\nu_{a,b}$ by

$$d\nu_{a,b}(\lambda) = w(a, b, \lambda) \rho(d\lambda).$$

2.3 Observed vs. unconditional correlations

Define

$$E_{\text{obs}}(a, b) := \int A(a, \lambda) B(b, \lambda) d\nu_{a,b}(\lambda) = \int A(a, \lambda) B(b, \lambda) w(a, b, \lambda) \rho(d\lambda),$$

and the unconditional (“full”) correlation

$$E_{\text{full}}(a, b) := \int A(a, \lambda) B(b, \lambda) \rho(d\lambda).$$

For a CHSH quartet (a, a', b, b') , define

$$S_{\text{obs}} := |E_{\text{obs}}(a, b) - E_{\text{obs}}(a, b')| + |E_{\text{obs}}(a', b) + E_{\text{obs}}(a', b')|.$$

2.4 A quantitative noncentrality statistic

Define the L^1 deviation from centrality

$$\delta(a, b) := \int |w(a, b, \lambda) - 1| \rho(d\lambda), \quad \delta := \sup_{a,b} \delta(a, b). \quad (2)$$

Note that $\text{TV}(\nu_{a,b}, \rho) = \frac{1}{2} \delta(a, b)$.

3 The central sector: where Bell–CHSH actually lives

Definition 1 (Central selection / fair sampling). Selection is *central* if $\eta_A(a, \lambda) \equiv \eta_A(a)$ and $\eta_B(b, \lambda) \equiv \eta_B(b)$. Equivalently, $w(a, b, \lambda) \equiv 1$ for all (a, b) .

Lemma 2 (Pointwise CHSH bound). *Fix λ . For any a, a', b, b' and any $A, B \in [-1, 1]$,*

$$|A(a, \lambda)B(b, \lambda) - A(a, \lambda)B(b', \lambda)| + |A(a', \lambda)B(b, \lambda) + A(a', \lambda)B(b', \lambda)| \leq 2.$$

Proof. This is the standard CHSH algebra; boundedness $|A|, |B| \leq 1$ is enough. $\square$

Theorem 3 (Centrality identity and Bell–CHSH). *Assume MI, locality, bounded outcomes, and central selection. Then $E_{\text{obs}}(a, b) = E_{\text{full}}(a, b)$ for all (a, b) , hence for any quartet $S_{\text{obs}} \leq 2$.*

Proof. If $w \equiv 1$ then $E_{\text{obs}} = E_{\text{full}}$ by definition. Lemma 2 holds pointwise in λ ; integrating w.r.t. ρ yields CHSH ≤ 2 . $\square$

Remark 4 (Hiddenness is irrelevant to CHSH). Lemma 2 is pointwise in λ . Whether λ is observed or not does not affect CHSH; what matters is whether the observed sample is governed by ρ (central) or by a reweighted $\nu_{a,b}$ (noncentral).

4 Noncentral selection: sharp inflation bounds

Theorem 5 (Deviation bound). *For each (a, b) ,*

$$|E_{\text{obs}}(a, b) - E_{\text{full}}(a, b)| \leq \delta(a, b).$$

Proof.

$$E_{\text{obs}} - E_{\text{full}} = \int AB(w - 1) d\rho,$$

so by $|AB| \leq 1$,

$$|E_{\text{obs}} - E_{\text{full}}| \leq \int |w - 1| d\rho = \delta(a, b).$$

$\square$

Theorem 6 (Sharp CHSH inflation). *Under MI and locality with bounded outcomes,*

$$S_{\text{obs}} \leq 2 + 4\delta.$$

Equivalently,

$$S_{\text{obs}} \leq 2 + 8 \sup_{a,b} \text{TV}(\nu_{a,b}, \rho).$$

Proof. Apply Theorem 5 to the four pairs in CHSH and use $S_{\text{full}} \leq 2$ (Theorem 3):

$$S_{\text{obs}} \leq S_{\text{full}} + \sum_{j=1}^4 \delta(\theta_j) \leq 2 + 4\delta.$$

$\square$

Corollary 7 (Tsirelson noncentrality threshold). *If $S_{\text{obs}} = 2\sqrt{2}$ is reproduced by a strictly local, MI model via selection, then necessarily*

$$\delta \geq \frac{2\sqrt{2} - 2}{4} \approx 0.2071.$$

5 Canonical RN-density recipe (measure-theoretic)

Fix a pair (a, b) . Suppose there exist measurable disjoint regions $U_{ab}^+, U_{ab}^- \subset \Lambda$ with $\rho(U_{ab}^\pm) > 0$ such that

$$A(a, \lambda)B(b, \lambda) = +1 \text{ on } U_{ab}^+, \quad A(a, \lambda)B(b, \lambda) = -1 \text{ on } U_{ab}^-.$$

Let $E^{\text{tgt}}(a, b) \in [-1, 1]$ be a target correlation and set

$$\alpha_{ab} := \frac{1 + E^{\text{tgt}}(a, b)}{2}.$$

Define the unnormalized RN density

$$\tilde{w}_{ab}(\lambda) := \frac{\alpha_{ab}}{\rho(U_{ab}^+)} \mathbf{1}_{U_{ab}^+}(\lambda) + \frac{1 - \alpha_{ab}}{\rho(U_{ab}^-)} \mathbf{1}_{U_{ab}^-}(\lambda) + \varepsilon \psi_{ab}(\lambda), \quad (3)$$

where $\psi_{ab} \geq 0$ has $\int \psi_{ab} d\rho = 1$ and $\varepsilon > 0$ is small. Normalize:

$$w_{ab}(\lambda) := \frac{\tilde{w}_{ab}(\lambda)}{\int \tilde{w}_{ab} d\rho}.$$

6 Explicit local factorized construction for arbitrary finite correlation tables

Fix finite settings $\{a_1, \dots, a_m\}$ and $\{b_1, \dots, b_n\}$ and targets $E_{ij}^{\text{tgt}} \in [-1, 1]$.

Theorem 8 (Finite-table simulation by strictly local detection selection). *There exists a probability space (Λ, ρ) , local deterministic outcomes $A(a_i, \lambda), B(b_j, \lambda) \in \{\pm 1\}$, and local detection indicators $D_A(a_i, \lambda), D_B(b_j, \lambda) \in \{0, 1\}$ such that:*

1. (MI) ρ is independent of (i, j) ,
2. (locality) A depends only on (i, λ) and B only on (j, λ) ,
3. the observed correlations conditioned on coincidence satisfy

$$E_{\text{obs}}(a_i, b_j) = E_{ij}^{\text{tgt}} \quad \text{for all } i, j.$$

Construction. Let $\Lambda := \{1, \dots, m\} \times \{1, \dots, n\} \times [0, 1]$ with ρ the product of uniform counting measure on $\{1, \dots, m\} \times \{1, \dots, n\}$ and Lebesgue on $[0, 1]$. Write $\lambda = (u, v, t)$.

Define local detection indicators

$$D_A(a_i, \lambda) := \mathbf{1}\{u = i\}, \quad D_B(b_j, \lambda) := \mathbf{1}\{v = j\}.$$

Then for setting pair (i, j) , coincidences occur iff $(u, v) = (i, j)$; the coincidence probability is $Z_{ij} = \rho(u = i, v = j) = 1/(mn)$.

Define outcomes

$$A(a_i, \lambda) := +1, \quad B(b_j, \lambda) := \begin{cases} +1, & t \leq \alpha_{ij}, \\ -1, & t > \alpha_{ij}, \end{cases} \quad \alpha_{ij} := \frac{1 + E_{ij}^{\text{tgt}}}{2}.$$

Conditioned on coincidence $(u, v) = (i, j)$, $t \sim \text{Unif}[0, 1]$, hence

$$\mathbb{E}[B \mid (u, v) = (i, j)] = (+1)\alpha_{ij} + (-1)(1 - \alpha_{ij}) = 2\alpha_{ij} - 1 = E_{ij}^{\text{tgt}}.$$

Since $A \equiv +1$ on coincidences, $E_{\text{obs}}(i, j) = E_{ij}^{\text{tgt}}$. $\square$

Remark 9. This is an explicit detection-loop-hole (noncentral selection) simulation. It does not contradict Bell's theorem because it violates centrality/fair-sampling. It also has coincidence rate $1/(mn)$; more efficient constructions exist for specific targets (e.g. singlet) but still require noncentrality quantified by δ .

7 PV-style motivation: fibered noninjectivity (optional)

A PV narrative can be formalized by assuming $\Lambda \cong V \times \Phi$ and a noninjective kinematic map $\mathcal{L} : V \times \Phi \rightarrow \mathcal{M}$: distinct (v, ϕ) share the same observed cell $\ell \in \mathcal{M}$ but differ internally. Then within a fixed observed cell, the auxiliary coordinate can carry opposite-sign subregions for AB , enabling reweighting without changing coarse observed kinematics.

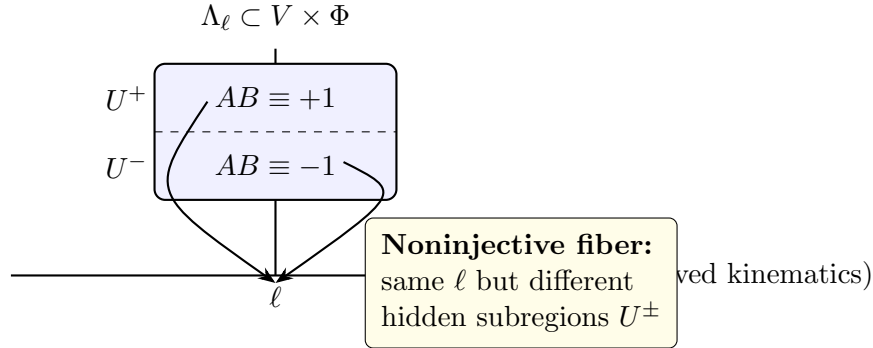


Figure 1: PV-style motivation: noninjective hidden fibers allow internal sign structure inside a fixed observed kinematic cell.

8 Experimental protocol to estimate/bound δ

The inflation bound $S_{\text{obs}} \leq 2 + 4\delta$ makes δ experimentally meaningful: if an experiment measures S_{obs} and independently upper-bounds δ below $(S_{\text{obs}} - 2)/4$, then all strictly local MI models of the noncentral-selection type are excluded.

Protocol overview (optics / time-tag experiments)

- P1. Trial definition (event-ready if possible).** Use heralding to define trials independent of outcomes; otherwise use fixed clock bins.
- P2. Measure singles efficiencies vs setting.** Estimate $\hat{\eta}_A(a)$ and $\hat{\eta}_B(b)$ for each setting; look for setting dependence.
- P3. Coincidence-window dithering.** If coincidences are defined by $|t_A - t_B| < W$, repeat for W on a small grid.
- P4. Threshold and spectral sweeps.** Repeat CHSH runs for detector thresholds $\pm 5\%$, $\pm 10\%$ and filters (e.g. 1,3,10 nm).
- P5. Auxiliary-tag binning (best practice).** Partition trials by an auxiliary tag X correlated with hidden microphysics (arrival-time residual, pulse energy, frequency bin, etc.) and estimate a binned w .
- P6. Compute a conservative upper confidence bound $\hat{\delta}^{(+)}$.** Compare S_{obs} to $2 + 4\hat{\delta}^{(+)}$ (one-sided test).

Concrete estimators

Window-derived bound. For each pair (a, b) define

$$\Delta E(a, b) := \max_W \hat{E}(a, b; W) - \min_W \hat{E}(a, b; W), \quad \hat{\delta}_W := \frac{1}{2} \max_{a, b} \Delta E(a, b).$$

Auxiliary-tag binned estimator (recommended). Let $X \in \{1, \dots, K\}$ be a tag with empirical frequencies $\hat{p}_i = \widehat{\Pr}(X = i)$. For each (a, b) and bin i record trials and coincidences and set

$$\hat{w}(a, b \mid i) := \frac{\widehat{\Pr}(\text{coinc} \mid a, b, X = i)}{\sum_{j=1}^K \widehat{\Pr}(\text{coinc} \mid a, b, X = j) \hat{p}_j}, \quad \hat{\delta}(a, b) := \sum_{i=1}^K |\hat{w}(a, b \mid i) - 1| \hat{p}_i, \quad \hat{\delta} := \max_{a, b} \hat{\delta}(a, b).$$

Bootstrap over trials to get an upper CI $\hat{\delta}^{(+)}$.

Decision rule. If

$$\hat{S}_{\text{obs}} > 2 + 4\hat{\delta}^{(+)} + z_{1-\alpha} \sigma_S,$$

then local MI noncentral-selection models with $\delta \leq \hat{\delta}^{(+)}$ are excluded at one-sided confidence $1 - \alpha$.

9 Conclusion

Bell-CHSH is correct in the central sector: if the detected sample is not a hidden-variable reweighting, then MI+locality imply $\text{CHSH} \leq 2$. What CHSH violation rules out depends operationally on whether centrality holds. The noncentral sector is parameterized by the RN weight w and its L^1 distance from centrality δ ; the sharp inflation bound $S_{\text{obs}} \leq 2 + 4\delta$ gives a quantitative bridge between theory and experimental audits.

A Appendix: a common mistaken “local model” and the correct integral

A frequently proposed local deterministic model is

$$A(\theta_a, \lambda) = \text{sgn}(\cos(\theta_a - \lambda)), \quad B(\theta_b, \lambda) = -\text{sgn}(\cos(\theta_b - \lambda)),$$

with $\lambda \sim \text{Unif}[0, 2\pi)$. Its correlation is not $-\cos(\theta_a - \theta_b)$. A direct computation yields a piecewise linear function of $\Delta = \theta_a - \theta_b \pmod{\pi}$:

$$E(\Delta) = -\left(1 - \frac{2|\Delta|}{\pi}\right),$$

so it cannot exceed $\text{CHSH} = 2$. Obtaining $-\cos$ locally requires relaxing centrality (selection dependence), or relaxing MI, or relaxing locality.